# NO UNWANTED UNIVERSALLY BAIRE MORPHISMS

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ABSTRACT. We show that the usual proof that there are no morphisms, whose constituent maps are Borel, between certain challenge-response relations generalizes to show that there are no morphisms whose constituent maps are universally Baire.

## 1. Morphisms

Let $\kappa$ be a cardinal. Recall that a set $A \subseteq {}^\omega\omega$ is *$\kappa$-universally Baire* (see [2]) iff there exist trees $T, S \subseteq {}^{<\omega}\omega \times {}^{<\omega}\delta$ for some cardinal $\delta$ such that $p[T] = A$ and in every forcing extension of $V$ by a forcing of size $\leq \kappa$, $p[T] = {}^\omega\omega - p[S]$. A set $A \subseteq {}^\omega\omega$ is *universally Baire* iff it is $\kappa$-universally Baire for all $\kappa$. We make a similar definition for relations on ${}^\omega\omega$ to be universally Baire. We say that a function is universally Baire iff its graph is. Given a set $A$ which is $\kappa$-universally Baire, witnessed by $T$ and $S$, and given a forcing of size $\leq \kappa$, we say that the set $p[T]$ (as computed in the extension) is what $A$ *lifts* to.

Let $\mathcal{R}_1 := \langle {}^\omega\omega, {}^\omega\omega, R_1\rangle$ and $\mathcal{R}_2 := \langle {}^\omega\omega, {}^\omega\omega, R_2\rangle$ be challenge-response relations (so $R_1, R_2 \subseteq {}^\omega\omega \times {}^\omega\omega$). It is natural to ask if there is a *morphism* form the first to the second. That is, a pair $\langle \phi_-, \phi_+\rangle$ of functions $\phi_-, \phi_+ : {}^\omega\omega \to {}^\omega\omega$ such that

$$(\forall x \in {}^\omega\omega)(\forall y \in {}^\omega\omega)\, \phi_-(x) R_1 y \Rightarrow x R_2 \phi_+(y).$$

In [1] (Theorem 4.15), a situation is given where there can be no such morphism with *either* $\phi_-$ or $\phi_+$ Borel. The goal of this document is to show why in the same situation it is impossible for *both* $\phi_-$ and $\phi_+$ to be universally Baire. This leaves open the question of whether one of $\phi_-$ or $\phi_+$ could be universally Baire..

Consider a challenge-response relation $\mathcal{R} = \langle {}^\omega\omega, {}^\omega\omega, R\rangle$ which has an interpretation in every forcing extension (this happens when the relation $R$ is universally Baire for example). Given a forcing $\mathbb{P}$, we say that that $\mathbb{P}$ is *$\mathcal{R}$-adequate* if

$$1_{\mathbb{P}} \Vdash (\forall x \in {}^\omega\omega)(\exists y \in {}^\omega\omega \cap \check{V})\, xRy.$$

**Lemma 1.1.** *Let $\kappa$ be an infinite cardinal. Let $f : {}^\omega\omega \to {}^\omega\omega$ be a function whose graph is $\kappa$-universally Baire. Then in every forcing extension by a poset of size $\leq \kappa$, $f$ lifts to a function defined on all of ${}^\omega\omega$.*

*Proof.* Let $\kappa$ be a cardinal and let $T, S$ be trees witnessing that the graph of $f$ is $\kappa$-universally Baire. Let $\mathbb{P}$ be a poset of size $\leq \kappa$. Let $G$ be $(V, \mathbb{P})$-generic. We want to show that $(p[T])^{V[G]}$ is the graph of a total function in $V[G]$. Given any $x \in ({}^\omega\omega)^{V[G]}$, we want some $y \in ({}^\omega\omega)^{V[G]}$ satisfying $(x, y) \in p[T]$.

Towards a contradiction, fix an $x \in ({}^\omega\omega)^{V[G]}$ such that there is no such corresponding $y$. From $T \subseteq {}^{<\omega}\omega \times {}^{<\omega}\omega \times {}^{<\omega}\delta$ we can form $T_x \subseteq {}^{<\omega}\omega \times {}^{<\omega}\delta$ which is a well-founded tree. Since $T_x$ is well-founded, it has some rank function $\sigma : T_x \to \omega_1$. Consider the tree $W$ whose nodes are pairs consisting of an element of ${}^{<\omega}\omega$ and a partial attempt to build a rank function for the corresponding tree from $T$. We have that $(x, \sigma)$ is a path through $W$. Since $W \in V$ and $W$ has a path in $V[G]$, it has a path in $V$. Such a path witnesses that $f$ is not total in $V$, which is a contradiction.

Using similar reasoning, it can be shown that in $V[G]$ there are not $x, y_1, y_2$ with $y_1 \notin y_2$ such that $(x, y_1) \in p[T]$ and $(x, y_2) \in p[T]$. $\square$

We will show the following. The hypothesis arises in practice, for example the proof that there is no Borel morphism from the splitting relation to the domination relation (see [1] Theorem 4.15).

**Proposition 1.2.** *Let $\mathcal{R}_1 = \langle {}^\omega\omega, {}^\omega\omega, R_1 \rangle$ and $\mathcal{R}_2 = \langle {}^\omega\omega, {}^\omega\omega, R_2 \rangle$ be challenge-response relations with $R_1$ and $R_2$ universally Baire. Suppose there is a forcing $\mathbb{P}$ which is $\mathcal{R}_1$-adequate but not $\mathcal{R}_2$-adequate. Then there is no morphism $\langle \phi_-, \phi_+ \rangle$ from $\mathcal{R}_1$ to $\mathcal{R}_2$ such that both the graph of $\phi_-$ and the graph of $\phi_+$ are universally Baire.*

*Proof.* Consider a pair of functions $\langle \phi_-, \phi_+ \rangle$ that are universally Baire. Let $\Phi$ be the statement that there exist $x_1, x_2, y_1, y_2 \in {}^\omega\omega$ satisfying the following

1) $\phi_-(x_1) = x_2$;
2) $\phi_+(y_1) = y_2$;
3) $x_2 R_1 y_1$;
4) $\neg x_1 R_2 y_2$.

$$\begin{tikzcd}
x_2 & R_1 & y_1 \arrow[d, "\phi_+"] \\
x_1 \arrow[u, "\phi_-"] & \neg R_2 & y_2.
\end{tikzcd}$$

We claim that $\Phi$ is equivalent to a statement which asserts the existence of a path through a tree in the ground model. The conditions 1)-4) are all of this form. For example, if $R_2 = p[T] = {}^\omega\omega - p[S]$, then 4) is equivalent to saying that $(x_1, y_2)$ is a path through $S$.

Note that by the definition of a morphism, if $\Phi$ holds, then $\langle \phi_-, \phi_+ \rangle$ is not a morphism from $\mathcal{R}_1$ to $\mathcal{R}_2$. Now since $\Phi$ is equivalent to a statement which asserts the existence of a path through a tree in the ground model, it is absolute between $V$ and forcing extensions. In particular, if we show that $\Phi$ holds after forcing with $\mathbb{P}$, then we are done.

Force with $\mathbb{P}$ to get $V[G]$. Let $x_1 \in ({}^\omega\omega)^{V[G]}$ witness that $\mathbb{P}$ is not $\mathcal{R}_2$-adequate. That is, there is no $y \in {}^\omega\omega \cap V$ satisfying $x_1 R_2 y$. By the lemma above, we may speak of $\phi_-(x_1)$. Since $\mathbb{P}$ is $\mathcal{R}_1$-adequate, let $y_1 \in {}^\omega\omega \cap V$ satisfy $\phi_-(x_1) R_1 y_1$. By what we said about $x_1$, we have $\neg x_1 R_2 \phi_+(y_1)$. Hence, $\Phi$ is satisfied. This completes the proof. $\square$

Note that this proposition says that $\phi_-$ and $\phi_+$ cannot *both* be universally Baire, whereas Theorem 4.15 of [1] says that *neither* $\phi_-$ nor $\phi_+$ can be Borel.

## 2. Weak Morphisms

There is a variant of the notion of morphism which is more general when one does not assume the Axiom of Choice. The idea is to replace functions with multiple valued functions.

**Definition 2.1.** Given challenge-response relations $\mathcal{A} = \langle {}^\omega\omega, {}^\omega\omega, A \rangle$ and $\mathcal{B} = \langle {}^\omega\omega, {}^\omega\omega, B \rangle$, a *weak morphism* from $\mathcal{A}$ to $\mathcal{B}$ is a pair $\langle F_-, F_+ \rangle$ of relations $F_-, F_+ \subseteq {}^\omega\omega \times {}^\omega\omega$ such that the following are satisfied:

1) $(\forall b_- \in {}^\omega\omega)(\exists a_- \in {}^\omega\omega)\, b_- F_- a_-$;
2) $(\forall a_+ \in {}^\omega\omega)(\exists b_+ \in {}^\omega\omega)\, a_+ F_+ b_+$;
3) for all $a_-, a_+, b_-, b_+ \in {}^\omega\omega$, if $b_- F_- a_-$, $a_+ F_+ b_+$, and $a_- A a_+$, then $b_- B b_+$.

This definition can be remembered by the following picture.

$$\begin{array}{ccc} a_- & A & a_+ \\ F_- & \Big\Downarrow & F_+ \\ b_- & B & b_+. \end{array}$$

If there is a morphism from $\mathcal{A}$ to $\mathcal{B}$, then there is a weak morphism from $\mathcal{A}$ to $\mathcal{B}$. Assuming the Axiom of Choice, the other direction holds as well. The proof of Proposition 1.2 can be easily modified to prove the corresponding result for weak morphisms.

## References

[1] Andreas Blass. Combinatorial Cardinal Characteristics of the Continuum. In M. Foreman and A. Kanamori, editors, *Handbook of Set Theory Volume 1*. Springer, New York, NY, 2010.

[2] Qi Feng, Menachem Magidor, and W. Hugh Woodin. Universally Baire sets of reals. In H. Judah, W. Just, and H. Woodin, editors, *Set Theory of the Continuum*, volume 26 of *Mathematical Sciences Research Institute Publications*, pages 203-242, Heidelberg, 1992. Stringer–Verlag.

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